A Version of Thirring's Approach to the KAM Theorem for Quadratic Hamiltonians with Degenerate Twist

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We give a proof of the KAM theorem on the existence of invariant tori for weakly perturbed Hamiltonian systems, based on Thirring's approach for Hamiltonians that are quadratic in the action variables. The main point of this approach is that the iteration of canonical transformations on which the proof is based stays within the space of quadratic Hamiltonians. We show that Thirring's proof for nondegenerate Hamiltonians can be adapted to Hamiltonians with degenerate twist. This case, in fact, drastically simplifies Thirring's proof.

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I. INTRODUCTION

Regular Hamiltonian dynamics is characterized by the existence of as many independent conserved quantities as degrees of freedom d. As a consequence, each trajectory is confined to evolve on an invariant torus of dimension d. The KAM technique was developed to prove the stability under small perturbations of a large fraction of these invariant tori. The KAM theorem [1–4] states that, if the frequency satisfies a diophantine condition, and the size of the perturbation ε is sufficiently small, then a torus of that frequency will be stable. The proof is based on an iterative algorithm to construct the invariant tori. Each step of the KAM iteration consists of a coordinate transformation that reduces the size of the perturbation from order ε to ε^2.

In this paper, we derive a KAM theorem for a family of Hamiltonians that has been used in Refs. [5–7] within the setup of a renormalization-group approach to the breakup of invariant tori.

We consider the following class of Hamiltonians with d degrees of freedom, that are quadratic in the action variables $\boldsymbol{A}=(A_1,A_2,\dots,A_d)\in R^d$ and described by three scalar functions of the angles $\boldsymbol{\varphi}=(\varphi_1,\varphi_2,\dots,\varphi_d)\in T^d$ (the d-dimensional torus parametrized, e. g. by $[0,2\pi]^d$):

$$H(\boldsymbol{A},\boldsymbol{\varphi})=\frac{1}{2}m(\boldsymbol{\varphi})(\boldsymbol{\Omega}\cdot\boldsymbol{A})^2 \\ +[\boldsymbol{\omega}_0+g(\boldsymbol{\varphi})\boldsymbol{\Omega}]\cdot\boldsymbol{A}+f(\boldsymbol{\varphi}) \ , \tag{1.1}$$

where $\boldsymbol{\omega}_0\in R^d$ is the frequency vector of the considered torus, and $\boldsymbol{\Omega}$ is some other constant vector. Without loss of generality, we assume that $\boldsymbol{\Omega}=(\Omega_1,\dots,\Omega_d)$ is of norm one, i. e. $|\boldsymbol{\Omega}|=\sum_{i=1}^d|\Omega_i|=1$. The functions m, g, and f are real analytic on T^d, i. e. they are holomorphic functions on some complex neighborhood of T^d.

We consider $\boldsymbol{\omega}_0$ verifying a diophantine condition

$$|\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}|^{-1}\le\sigma|\boldsymbol{\nu}|^{\tau},\ \forall\boldsymbol{\nu}=(\nu_1,\dots,\nu_d)\in Z^d\setminus\{0\}, \tag{1.2}$$

for some $\tau>d-1$ and $\sigma>0$.

The aim is to prove the existence of a torus with frequency vector $\boldsymbol{\omega}_0$ for Hamiltonian systems described by Eq. (1.1). The method is to find a canonical transformation such that the equations of motion expressed in the new coordinates show trivially the existence of this torus. For Hamiltonians of type (1.1), if one takes g and f equal to zero, then the resulting equations of motion are

$$\frac{d\boldsymbol{A}}{dt}=-\frac{1}{2}\frac{\partial m}{\partial\boldsymbol{\varphi}}(\boldsymbol{\Omega}\cdot\boldsymbol{A})^2, \tag{1.3}$$

$$\frac{d\boldsymbol{\varphi}}{dt}=m(\boldsymbol{\varphi})(\boldsymbol{\Omega}\cdot\boldsymbol{A})\boldsymbol{\Omega}+\boldsymbol{\omega}_0. \tag{1.4}$$

Thus, there exists a torus with frequency vector $\boldsymbol{\omega}_0$ located at $\boldsymbol{A}=0$ (even if the resulting Hamiltonian is not globally integrable, i. e. for $\boldsymbol{A}\neq 0$). This canonical transformation cannot be defined directly, because the formal expressions that appear in the classical perturbation theory do not converge due to the presence of small denominators. The construction is done via an iterative algorithm. We iterate a canonical change of coordinates that maps a Hamiltonian of the form (1.1) with a perturbation (g,f) of order $\mathcal{O}(\varepsilon)$, into a Hamiltonian with (g',f') of order $\mathcal{O}(\varepsilon^2)$. When the iteration converges, the perturbation (g,f) is completely eliminated.

The fact that m does not need to be eliminated to prove the existence of the torus with frequency vector $\boldsymbol{\omega}_0$, allows us to stay with Hamiltonians that are quadratic in the actions at each step of the iteration. This approach has been developed by Thirring [8,9]. In fact, Thirring considered a non-degenerate family of Hamiltonians quadratic in the actions, of the form

$$H(\boldsymbol{A},\boldsymbol{\varphi})=\frac{1}{2}\boldsymbol{A}\cdot M(\boldsymbol{\varphi})\boldsymbol{A} \\ +[\boldsymbol{\omega}_0+\boldsymbol{g}(\boldsymbol{\varphi})]\cdot\boldsymbol{A}+f(\boldsymbol{\varphi}) \ , \tag{1.5}$$

where M is a $d\times d$ matrix such that det $M\neq 0$, and $\boldsymbol{g}$ a vector. This implies that Thirring's Hamiltonian (1.5) satisfies the "twist condition"

$$\det\left|\frac{\partial^2 H}{\partial\boldsymbol{A}\partial\boldsymbol{A}}\right|=\det M\neq 0. \tag{1.6}$$

The Hamiltonian (1.1) does not satisfy the twist condition for any $\boldsymbol{\varphi}$. The extension of the KAM theorem for degenerate systems was done by Arnold [10] (see also Refs. [11–15]). In the case we consider here, the rank

of the matrix in (1.6) is one. Thus, there is a twist in a particular direction, which is the only relevant one for the considered perturbation.

Condition (1.6) is also called (standard) nondegeneracy condition. KAM theorems were also proven for Hamiltonians which satisfy the isoenergetic nondegeneracy condition [16,17], for which the following determinant of order $d+1$ does not vanish

$$\det \begin{vmatrix} \dfrac{\partial^2 H}{\partial \boldsymbol{A} \partial \boldsymbol{A}} & \dfrac{\partial H}{\partial \boldsymbol{A}} \\ \left(\dfrac{\partial H}{\partial \boldsymbol{A}}\right)^T & 0 \end{vmatrix} \neq 0. \tag{1.7}$$

For $d = 2$, the determinant (1.7) is equal to $-m(\boldsymbol{\varphi})[\det\,(\boldsymbol{\Omega}, \boldsymbol{\omega}_0)]^2$. Thus, the isoenergetic nondegeneracy condition is not satisfied if $m(\boldsymbol{\varphi})$ has zeroes.

For $d > 2$, the model (1.1) is isoenergetically degenerate. The present result is thus not a direct consequence of the ones in [16,17]. We notice that in the case where $\boldsymbol{\Omega}$ is parallel to $\boldsymbol{\omega}_0$, the Hamiltonian (1.1) is integrable (because of the existence of d integrals of motion in involution : H and $\boldsymbol{\omega}_0^{\perp} \cdot \boldsymbol{\varphi}$, where $\boldsymbol{\omega}_0^{\perp}$ denotes a vector perpendicular to $\boldsymbol{\omega}_0$, and there are $d-1$ such independant vectors).

In this article, we present a self-contained proof of the KAM theorem for Hamiltonian (1.1) based on Thirring's proof for Hamiltonian (1.5). The advantages are twofold: On one hand, we show that Thirring's proof can be adapted to degenerate twist Hamiltonians, and on the other hand, the resulting proof becomes even simpler than Thirring's. The fact that the iteration stays within the space of Hamiltonians quadratic in the actions is very useful, e. g., for numerical implementations to study the breakup of invariant tori [5–7].

Before entering into details of the theorem, we give basic definitions and notations: We denote $\mathcal{D}_\rho$ a complex neighborhood of T^d defined by

$$\mathcal{D}_\rho = \{\boldsymbol{\varphi} \in C^d | \quad \|\text{Im}\ \boldsymbol{\varphi}\| \leq \rho\}, \tag{1.8}$$

where $\|z\| = \max_i(|z_i|)$, for any $z \in C^d$. We will consider, in the following calculations, scalar functions defined in $\mathcal{D}_\rho$. More precisely, we define $\mathcal{A}_\rho$ as the set of complex functions $f(\boldsymbol{\varphi})$ defined on $\mathcal{D}_\rho$, analytic in the interior of $\mathcal{D}_\rho$, of period 2π in the variables φ_i, and which have real values when $\boldsymbol{\varphi} \in R^d$. We define a norm on $\mathcal{A}_\rho$: $\|f\|_\rho = \sup_{\boldsymbol{\varphi} \in \mathcal{D}_\rho} |f(\boldsymbol{\varphi})|$. We define $\langle f \rangle$, the mean value of f by

$$\langle f \rangle = \int_{T^d} \frac{d^d\boldsymbol{\varphi}}{(2\pi)^d} f(\boldsymbol{\varphi}). \tag{1.9}$$

In the following sections, we will use the notation $\boldsymbol{\partial} f = \dfrac{\partial f}{\partial \boldsymbol{\varphi}}$ for any function of the angles.

In Sec. II, we define the KAM iteration. In Sec. III, we give estimates on the transformed functions. Finally, in Sec. IV, we iterate the transformation, and prove the following KAM theorem for the family of Hamiltonians (1.1):

Theorem 1 *For* $H(\boldsymbol{A}, \boldsymbol{\varphi}) = \dfrac{1}{2} m(\boldsymbol{\varphi})(\boldsymbol{\Omega} \cdot \boldsymbol{A})^2 + [\boldsymbol{\omega}_0 + g(\boldsymbol{\varphi})\boldsymbol{\Omega}] \cdot \boldsymbol{A} + f(\boldsymbol{\varphi})$, *suppose that*

(i) $|\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}|^{-1} \leq \sigma |\boldsymbol{\nu}|^\tau$, $\forall \boldsymbol{\nu} \in Z^d \setminus \{0\}$, *for some* $\sigma > 0$ *and* $\tau > d-1$*;*

(ii) m, g, *and* f *are analytic in* $\mathcal{D}_\rho$, *and* $|\boldsymbol{\Omega}| = 1$*;*

(iii) $\max(\|g\|_\rho, \|f\|_\rho) \leq \beta_0$, *where*

$$\begin{aligned} &\beta_0 = 2^{-10}\Gamma^{-6} c^{-3} (h/3)^{3(\tau+d+1)}, \\ &c = 2^{3d} \sigma \left[2e^{-1}(\tau+1)\right]^{\tau+1}, \\ &\Gamma = \max(1, \|m\|_\rho, |\langle m \rangle|^{-1}), \\ &h < \inf(2\rho/9, c^{1/(\tau+d+1)}); \end{aligned}$$

Then, there exists a canonical transformation, analytic in $\mathcal{D}_{\rho-9h/2}$, *such that the Hamiltonian expressed in the new coordinates is*

$$H^{(\infty)} = \frac{1}{2} m^{(\infty)}(\boldsymbol{\varphi}^{(\infty)})(\boldsymbol{\Omega} \cdot \boldsymbol{A}^{(\infty)})^2 + \boldsymbol{\omega}_0 \cdot \boldsymbol{A}^{(\infty)},$$

where $m^{(\infty)}$ *is analytic in* $\mathcal{D}_{\rho-9h/2}$. *As a consequence, the system has an analytic invariant torus of frequency* $\boldsymbol{\omega}_0$.

Remark: if the constant part $\langle m \rangle$ of the quadratic term is zero, the maximal amplitude of the perturbation β_0 given by the theorem becomes zero. Thus, in order to have a nontrivial result, we require that $\langle m \rangle$ is nonzero.

II. EXPRESSIONS OF THE GENERATING FUNCTION AND OF THE NEW HAMILTONIAN

We perform a canonical transformation $\mathcal{U}_F : (\boldsymbol{\varphi}, \boldsymbol{A}) \mapsto (\boldsymbol{\varphi}', \boldsymbol{A}')$ defined by a generating function [18,19] linear in the action variables, and characterized by two scalar functions Y, Z, of the angles, and a constant a, of the form

$$F(\boldsymbol{A}', \boldsymbol{\varphi}) = (\boldsymbol{A}' + a\boldsymbol{\Omega}) \cdot \boldsymbol{\varphi} + Y(\boldsymbol{\varphi}) \boldsymbol{\Omega} \cdot \boldsymbol{A}' + Z(\boldsymbol{\varphi}), \tag{2.1}$$

leading to

$$\boldsymbol{A} = \frac{\partial F}{\partial \boldsymbol{\varphi}} = \boldsymbol{A}' + (\boldsymbol{\Omega} \cdot \boldsymbol{A}')\, \boldsymbol{\partial} Y + a\boldsymbol{\Omega} + \boldsymbol{\partial} Z, \tag{2.2}$$

$$\boldsymbol{\varphi}' = \frac{\partial F}{\partial \boldsymbol{A}'} = \boldsymbol{\varphi} + Y(\boldsymbol{\varphi})\boldsymbol{\Omega}. \tag{2.3}$$

Inserting Eq. (2.2) into the Hamiltonian (1.1), we obtain the expression of the Hamiltonian in the mixed representation of new action variables and old angle variables

$$\begin{aligned} \tilde{H}(\boldsymbol{A}', \boldsymbol{\varphi}) = &\frac{1}{2} \tilde{m}(\boldsymbol{\varphi})(\boldsymbol{\Omega} \cdot \boldsymbol{A}')^2 \\ &+ [\boldsymbol{\omega}_0 + \tilde{g}(\boldsymbol{\varphi})\boldsymbol{\Omega}] \cdot \boldsymbol{A}' + \tilde{f}(\boldsymbol{\varphi})\ , \end{aligned} \tag{2.4}$$

with

$$\tilde{m} = (1 + \mathbf{\Omega} \cdot \boldsymbol{\partial} Y)^2 m, \tag{2.5}$$
$$\tilde{g} = g + \boldsymbol{\omega}_0 \cdot \boldsymbol{\partial} Y + mb + \mathbf{\Omega} \cdot \boldsymbol{\partial} Y \left(g + mb\right), \tag{2.6}$$
$$\tilde{f} = f + \boldsymbol{\omega}_0 \cdot \boldsymbol{\partial} Z + \frac{1}{2} mb^2 + gb, \tag{2.7}$$

where $b(\boldsymbol{\varphi}) = a\Omega^2 + \mathbf{\Omega} \cdot \boldsymbol{\partial} Z$. As the new angles do not depend on the actions, the Hamiltonian (1.1) expressed in the new variables is also quadratic in the actions, and of the same form as (1.1). We notice that this transformation does not change $\mathbf{\Omega}$.
We determine the generating function (2.1) such that the new functions $\tilde{g}$, $\tilde{f}$ in $\mathcal{H} \circ \mathcal{U}_F$ vanish to the first order in ε. This leads to the conditions

$$\boldsymbol{\omega}_0 \cdot \boldsymbol{\partial} Z + f = const, \tag{2.8}$$
$$\boldsymbol{\omega}_0 \cdot \boldsymbol{\partial} Y + g + m\left(a\Omega^2 + \mathbf{\Omega} \cdot \boldsymbol{\partial} Z\right) = 0. \tag{2.9}$$

The constant a allows us to have $\langle \tilde{g} \rangle = \mathcal{O}(\varepsilon^2)$, in order to keep the frequency $\boldsymbol{\omega}_0$ at the chosen value. We recall that the functions g and f are of order $\mathcal{O}(\varepsilon)$ and m is of order one; as a consequence Y, Z and a are of order $\mathcal{O}(\varepsilon)$. Equations (2.8) and (2.9) are solved by representing them in Fourier space. They define the generating function F as

$$Z(\boldsymbol{\varphi}) = \sum_{\nu \neq 0} \frac{i}{\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}} f_\nu e^{i\nu\cdot\varphi}, \tag{2.10}$$

$$a = -\frac{\langle g \rangle + \langle m \mathbf{\Omega} \cdot \boldsymbol{\partial} Z \rangle}{\Omega^2 \langle m \rangle}, \tag{2.11}$$

$$Y(\boldsymbol{\varphi}) = \sum_{\nu \neq 0} \frac{i}{\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}} \big(g_\nu + (m\mathbf{\Omega} \cdot \boldsymbol{\partial} Z)_\nu + m_\nu a \Omega^2\big) e^{i\nu\cdot\varphi}, \tag{2.12}$$

where the scalar functions m, g, f are represented by their Fourier series, e. g.

$$f(\boldsymbol{\varphi}) = \sum_{\nu \in Z^d} f_\nu e^{i\nu\cdot\varphi}. \tag{2.13}$$

Thus the scalar functions in $\tilde{H}$ become

$$\tilde{m} = (1 + \mathbf{\Omega} \cdot \boldsymbol{\partial} Y)^2 m, \tag{2.14}$$
$$\tilde{g} = -\left(\boldsymbol{\omega}_0 \cdot \boldsymbol{\partial} Y\right)\left(\mathbf{\Omega} \cdot \boldsymbol{\partial} Y\right), \tag{2.15}$$
$$\tilde{f} = \frac{1}{2}(g - \boldsymbol{\omega}_0 \cdot \boldsymbol{\partial} Y)\left(a\Omega^2 + \mathbf{\Omega} \cdot \boldsymbol{\partial} Z\right). \tag{2.16}$$

The expression of $\tilde{H}$ in the new angles requires the inversion of Eq. (2.3). We denote H' the Hamiltonian expressed in the new variables $H'(\boldsymbol{A}', \boldsymbol{\varphi}') = \tilde{H}(\boldsymbol{A}', \boldsymbol{\varphi})$:

$$H'(\boldsymbol{A}', \boldsymbol{\varphi}') = \frac{1}{2} m'(\boldsymbol{\varphi}')(\mathbf{\Omega} \cdot \boldsymbol{A}')^2 + [\boldsymbol{\omega}_0 + g'(\boldsymbol{\varphi}')\mathbf{\Omega}] \cdot \boldsymbol{A}' + f'(\boldsymbol{\varphi}') \,. \tag{2.17}$$

In the following section, we give estimates on Z, Y and a, in order to derive estimates on m', g' and f'.

III. ESTIMATES ON THE GENERATING FUNCTION AND ON THE NEW HAMILTONIAN

A. Estimate on the generating function

Lemma 1 *Let f be element of $\mathcal{A}_\rho$, and $\boldsymbol{\omega}_0$ satisfy Eq. (1.2), then $Z(\boldsymbol{\varphi})$ given by Eq. (2.10) is analytic on $\mathcal{D}_{\rho-h}$ (for any choice of h with $0 < h < \rho < 2$), and satisfies the following estimates*

$$\|Z\|_{\rho-h} \leq \bar{c} h^{-\tau-d} \|f\|_\rho, \tag{3.1}$$
$$\|\boldsymbol{\omega}_0 \cdot \boldsymbol{\partial} Z\|_{\rho-h} \leq \|f\|_\rho, \tag{3.2}$$
$$\|\mathbf{\Omega} \cdot \boldsymbol{\partial} Z\|_{\rho-h} \leq c h^{-\tau-d-1} \|f\|_\rho, \tag{3.3}$$

where $\bar{c} = 2^{3d}\sigma\left(2e^{-1}\tau\right)^\tau$ *and* $c = 2^{3d}\sigma\left[2e^{-1}(\tau+1)\right]^{\tau+1}$.

Proof:
This lemma is proved, for instance, in Refs. [10,20].
First, we estimate the Fourier coefficients of f expressed as integrals of f over the torus,

$$f_\nu = \int_{T^d} \frac{d^d\boldsymbol{\varphi}}{(2\pi)^d} f(\boldsymbol{\varphi}) e^{-i\nu\cdot\varphi}. \tag{3.4}$$

By a shift of the angles $\boldsymbol{\varphi} \mapsto \boldsymbol{\varphi} + i\boldsymbol{\eta}$, where $\eta_i = -\rho\nu_i/|\nu_i|$,

$$f_\nu = \int_{T^d} \frac{d^d\boldsymbol{\varphi}}{(2\pi)^d} f(\boldsymbol{\varphi} + i\boldsymbol{\eta}) e^{-i\nu\cdot\varphi} e^{-\rho|\nu|}. \tag{3.5}$$

Then $|f_\nu| \leq \|f\|_\rho e^{-\rho|\nu|}$, for all $\boldsymbol{\nu} \in Z^d$.
To estimate Z given by Eq. (2.10), we use the diophantine property (1.2),

$$|Z(\boldsymbol{\varphi})| \leq \sigma \sum_{\nu \neq 0} |\boldsymbol{\nu}|^\tau |f_\nu| e^{-\nu\cdot\mathrm{Im}\varphi}. \tag{3.6}$$

Then, for all $\boldsymbol{\varphi} \in \mathcal{D}_{\rho-h}$, we have

$$|Z(\boldsymbol{\varphi})| \leq \sigma \|f\|_\rho \sum_{\nu \neq 0} |\boldsymbol{\nu}|^\tau e^{-h|\nu|}. \tag{3.7}$$

To estimate the sum, we use the following property which is easy to check (see Ref. [2]):

$$|\boldsymbol{\nu}|^\tau \leq \left(\frac{2\tau}{eh}\right)^\tau e^{|\nu|h/2}, \quad \forall \tau, h > 0. \tag{3.8}$$

From the fact that $1/(1 - e^{-x}) < 2/x$, for all $x \in]0,1[$, we have

$$\sum_{\nu \neq 0} e^{-|\nu|h/2} < \left(\frac{8}{h}\right)^d, \tag{3.9}$$

which gives the estimate (3.1). The estimate (3.3) is obtained by the same calculations, and the estimate (3.2) is straightforward from Eq. (2.8). □

Lemma 2 *Let m, g, and f be elements of $\mathcal{A}_\rho$, and $\boldsymbol{\omega}_0$ satisfy Eq. (1.2), then $Y(\boldsymbol{\varphi})$ given by Eq. (2.12) is analytic on $\mathcal{D}_{\rho-2h}$. The constant a and the function Y satisfy the following estimates*

$$\|Y\|_{\rho-2h} \le \bar{c}h^{-\tau-d}\big(1+|\langle m\rangle|^{-1}\|m\|_\rho\big) \times \left(\|g\|_\rho + ch^{-\tau-d-1}\|f\|_\rho\,\|m\|_\rho\right), \quad (3.10)$$

$$\|\boldsymbol{\omega}_0\cdot\boldsymbol{\partial}Y\|_{\rho-2h} \le \big(1+|\langle m\rangle|^{-1}\|m\|_\rho\big) \times \left(\|g\|_\rho + ch^{-\tau-d-1}\|f\|_\rho\,\|m\|_\rho\right), \quad (3.11)$$

$$\|\boldsymbol{\Omega}\cdot\boldsymbol{\partial}Y\|_{\rho-2h} \le ch^{-\tau-d-1}\left(1+|\langle m\rangle|^{-1}\|m\|_\rho\right) \times \left(\|g\|_\rho + ch^{-\tau-d-1}\|f\|_\rho\,\|m\|_\rho\right), \quad (3.12)$$

$$|a|\Omega^2 \le |\langle m\rangle|^{-1}\left(\|g\|_\rho + ch^{-\tau-d-1}\|f\|_\rho\,\|m\|_\rho\right). \quad (3.13)$$

The proof is the same as the one for the estimates on Z.

B. Estimate on the new Hamiltonian

The expression of the scalar functions of the Hamiltonian expressed in the new actions and old angles are explicitly known, and estimates on these functions are easy to obtain from the estimates on the generating function. One has then to invert Eq. (2.3) in order to obtain the estimate on the Hamitonian expressed in the new angle variables. The Jacobian of this transformation is

$$\det\left|\frac{\partial\varphi'_{\mathrm{j}}}{\partial\varphi_{\mathrm{k}}}\right| = 1+\boldsymbol{\Omega}\cdot\boldsymbol{\partial}\mathrm{Y}. \quad (3.14)$$

In this section, we denote $V_\rho = \max(\|f\|_\rho, \|g\|_\rho)$ and $\Gamma = \max(1, \|m\|_\rho, |\langle m\rangle|^{-1})$.
From Eq. (2.3) and estimate (3.10), one has the following inequality:

$$|\boldsymbol{\varphi}'-\boldsymbol{\varphi}| \le 2\Gamma^3 ch^{-\tau-d}\left(1+ch^{-\tau-d-1}\right)V_\rho, \quad (3.15)$$

for all $\boldsymbol{\varphi}\in\mathcal{D}_{\rho-2h}$ (we recall that $\Gamma\ge 1$). If we assume that V_ρ and h are sufficiently small, one has an estimate on the new angles. More precisely, we assume the following inequalities:

$$\Gamma^3c^2h^{-2(\tau+d+1)}V_\rho \le \frac{1}{4}, \quad (3.16)$$

$$ch^{-\tau-d-1} \ge 1. \quad (3.17)$$

Then, for all $\boldsymbol{\varphi}\in\mathcal{D}_{\rho-2h}$, we have

$$|\boldsymbol{\varphi}'-\boldsymbol{\varphi}(\boldsymbol{\varphi}')| \le h. \quad (3.18)$$

As a consequence, $\mathcal{D}_{\rho-2h}\subset\Phi(\mathcal{D}_{\rho-3h})$ by the map $\boldsymbol{\varphi}'\mapsto\boldsymbol{\varphi}=\Phi(\boldsymbol{\varphi}')$ given by Eq. (2.3). In order to express the estimates with respect to $\boldsymbol{\varphi}'$, it suffices thus to restrict the width of the strip $\mathcal{D}$ from $\rho-2h$ to $\rho-3h$, e. g. $\|f'\|_{\rho-3h}\le\|\tilde{f}\|_{\rho-2h}$. Then, the estimates on the new perturbation (g', f') are obtained from Eqs. (2.15)-(2.16) and from the estimates on the generating function:

$$\|g'\|_{\rho-3h} \le 2^4\Gamma^6c^3h^{-3(\tau+d+1)}V_\rho^2, \quad (3.19)$$

$$\|f'\|_{\rho-3h} \le 2^3\Gamma^5c^2h^{-2(\tau+d+1)}V_\rho^2. \quad (3.20)$$

Taking into account condition (3.17), we obtain the estimate on $V'_{\rho-3h} = \max(\|g'\|_{\rho-3h}, \|f'\|_{\rho-3h})$:

$$V'_{\rho-3h} \le 2^4\Gamma^6c^3h^{-3(\tau+d+1)}V_\rho^2. \quad (3.21)$$

Concerning the quadratic term, we deduce from Eqs. (3.12) and (2.14) that

$$\|m'\|_{\rho-3h} \le \|m\|_\rho\left(1+4\Gamma^3c^2h^{-2(\tau+d+1)}V_\rho\right)^2. \quad (3.22)$$

Then using Hypothesis (*iii*) of the theorem, we obtain $\|m'\|_{\rho-3h}\le 2\Gamma$.
The mean value of m' is determined by the integral

$$\langle m'\rangle = \int_{T^d}\frac{d^d\boldsymbol{\varphi}'}{(2\pi)^d}m'(\boldsymbol{\varphi}'). \quad (3.23)$$

With the change of variable $\boldsymbol{\varphi}'\mapsto\boldsymbol{\varphi}$ given by Eqs. (2.3)-(3.14) and using Eq. (2.14), we rewrite the integral

$$\langle m'\rangle = \int_{T^d}\frac{d^d\boldsymbol{\varphi}}{(2\pi)^d}|1+\boldsymbol{\Omega}\cdot\boldsymbol{\partial}Y|\tilde{m}(\boldsymbol{\varphi}) = \int_{T^d}\frac{d^d\boldsymbol{\varphi}}{(2\pi)^d}|1+\boldsymbol{\Omega}\cdot\boldsymbol{\partial}Y|^3 m(\boldsymbol{\varphi}). \quad (3.24)$$

To estimate $|\langle m'\rangle|^{-1}$, we first estimate the difference $|\langle m'\rangle-\langle m\rangle|$ using Eq. (3.24), and we obtain

$$|\langle m'\rangle-\langle m\rangle| \le \|m\|_\rho\big(3\|\boldsymbol{\Omega}\cdot\boldsymbol{\partial}Y\|_{\rho-2h} + 3\|\boldsymbol{\Omega}\cdot\boldsymbol{\partial}Y\|^2_{\rho-2h} + \|\boldsymbol{\Omega}\cdot\boldsymbol{\partial}Y\|^3_{\rho-2h}\big). \quad (3.25)$$

From Eq. (3.12) and condition (3.16), we have the estimate

$$\|\boldsymbol{\Omega}\cdot\boldsymbol{\partial}Y\|_{\rho-2h} \le 4\Gamma^3c^2h^{-2(\tau+d+1)}V_\rho \le 1, \quad (3.26)$$

which leads to

$$|\langle m'\rangle-\langle m\rangle| \le 28\Gamma^4c^2h^{-2(\tau+d+1)}V_\rho. \quad (3.27)$$

One can easily check that from Hypothesis (*iii*), it follows that

$$|\langle m'\rangle-\langle m\rangle| \le \frac{1}{2\Gamma}. \quad (3.28)$$

Writing that $\langle m'\rangle = \langle m\rangle + \langle m'\rangle - \langle m\rangle$, we have the following estimate on $|\langle m'\rangle|^{-1}$:

$$|\langle m'\rangle|^{-1} \le \frac{|\langle m\rangle|^{-1}}{|1-|\langle m\rangle|^{-1}|\langle m'\rangle-\langle m\rangle||}. \quad (3.29)$$

From Eq. (3.28), we deduce that $|\langle m'\rangle|^{-1}\le 2\Gamma$. Therefore, we have proved that $\Gamma'\equiv\max(1, \|m'\|_{\rho-3h}, |\langle m'\rangle|^{-1})\le 2\Gamma$.

IV. CONVERGENCE OF THE ITERATION: KAM THEOREM

The estimate (3.21) gives a precise meaning to the statement that the perturbation is reduced from V to V^2: we see that the actual reduction is somewhat smaller, since h depends on V_ρ through the condition (3.16) [the actual rate of reduction that takes this into account will be expressed by Eq. (4.11) below]. The counterpart is that the width of the domain of analyticity of the new functions is reduced from ρ to $\rho-3h$; for this reason, h is called the "analyticity loss parameter". At the nth step of the iteration, we choose h_n as the analyticity loss parameter such that the final domain of analyticity (after an infinite number of iterations) does not shrink to zero. For instance, we choose

$$h_n = h3^{-n+1}, \quad \text{with } h < \inf(2\rho/9, c^{1/(\tau+d+1)}). \quad (4.1)$$

The second term in the inf is to guarantee also the condition
(3.17). We denote $\Gamma_n = \max(1, \|m^{(n)}\|_{\rho_n}, |\langle m^{(n)}\rangle|^{-1})$ and $V_n = \max(\|g^{(n)}\|_{\rho_n}, \|f^{(n)}\|_{\rho_n})$, where $m^{(n)}$, $g^{(n)}$ and $f^{(n)}$ denote the three scalar functions defining the Hamiltonian (1.1) after n iterations, and

$$\rho_n = \rho_{n-1} - 3h_n = \rho - 3\sum_{k=1}^{n} h_k. \quad (4.2)$$

The previous section gave the following estimates

$$V_n \leq 2^4 \Gamma_{n-1}^6 c^3 h_n^{-3(\tau+d+1)} V_{n-1}^2, \quad (4.3)$$

$$\|m^{(n)}\|_{\rho_n} \leq \|m^{(n-1)}\|_{\rho_{n-1}} \times \left(1 + 4\Gamma_{n-1}^3 c^2 h_n^{-2(\tau+d+1)} V_{n-1}\right)^2. \quad (4.4)$$

We now prove that Γ_n is bounded, and that V_n tends to zero faster than geometrically as n goes to infinity.
Denoting

$$V_0 = \max(\|g\|_\rho, \|f\|_\rho), \quad (4.5)$$
$$\Gamma = \max(1, \|m\|_\rho, |\langle m\rangle|^{-1}), \quad (4.6)$$
$$\gamma = 2^{10}\Gamma^6 c^3 h^{-3(\tau+d+1)}, \quad (4.7)$$
$$\delta = 3^{3(\tau+d+1)}, \quad (4.8)$$

we define the sequence of values

$$\varepsilon_n = \gamma\delta^{n-1}\varepsilon_{n-1}^2 = \frac{(\gamma\delta V_0)^{2^n}}{\gamma\delta^{n+1}}. \quad (4.9)$$

We will show by induction that

$$\Gamma_k \leq 2\Gamma, \quad (4.10)$$
$$V_k \leq \varepsilon_k, \quad (4.11)$$

for all k. If we assume V_0 to be sufficiently small, such that $\gamma\delta V_0 < 1$, i. e.

$$2^{10}\Gamma^6 c^3 h^{-3(\tau+d+1)} 3^{3(\tau+d+1)} V_0 < 1, \quad (4.12)$$

which is the hypothesis (iii) of the theorem, then V_n tends to zero faster than $\delta^{-n} = 3^{-3n(\tau+d+1)}$ as n goes to infinity. The bounds (4.10)-(4.11) are satisfied for $k=0$, since $\Gamma_0 = \Gamma > 0$ and $V_0 = \varepsilon_0$.
Suppose that for all $k < n$, these bounds are satisfied. From Eqs. (4.3) and (4.10), we deduce that

$$V_n \leq \gamma\delta^{n-1} V_{n-1}^2. \quad (4.13)$$

Then using Eq. (4.11), we have $V_n \leq \varepsilon_n$. Concerning Γ_n, Eq. (4.4) leads to

$$\|m^{(n)}\|_{\rho_n} \leq \|m\|_\rho \times \prod_{k=0}^{n-1} \left(1 + 2^5\Gamma^3 c^2 h^{-2(\tau+d+1)} 3^{2k(\tau+d+1)} V_k\right)^2.$$

Using Eq. (4.11) and Hypothesis (iii), we obtain

$$2^5\Gamma^3 c^2 h^{-2(\tau+d+1)} 3^{2k(\tau+d+1)} V_k \leq 3^{-(k+3)(\tau+d+1)}. \quad (4.14)$$

Then we have

$$\|m^{(n)}\|_{\rho_n} \leq \Gamma \prod_{k=0}^{\infty} \left(1 + 3^{-(k+3)(\tau+d+1)}\right)^2. \quad (4.15)$$

Using the fact that $\prod(1+x_k)^2 \leq \exp 2\sum x_k$, we show that the infinite product converges, and that it is smaller than 2. Thus, $\|m^{(n)}\|_{\rho_n} \leq 2\Gamma$.
For $|\langle m^{(n)}\rangle|^{-1}$, we use estimate (3.29):

$$\frac{|\langle m^{(n)}\rangle|^{-1}}{|\langle m^{(n-1)}\rangle|^{-1}} \leq \frac{1}{\left|1 - |\langle m^{(n-1)}\rangle|^{-1} |\langle m^{(n)}\rangle - \langle m^{(n-1)}\rangle|\right|}. \quad (4.16)$$

The difference $|\langle m^{(n)}\rangle - \langle m^{(n-1)}\rangle|$ is evaluated as in the first step (3.27):

$$|\langle m^{(n)}\rangle - \langle m^{(n-1)}\rangle| \leq 28 \cdot 2^4 \Gamma^4 c^2 h^{-2(\tau+d+1)} \times 3^{2(n-1)(\tau+d+1)} V_{n-1}. \quad (4.17)$$

Using Eq. (4.11), Hypothesis (iii), and the fact that Γ and $ch^{-(\tau+d+1)}$ are greater than one,

$$|\langle m^{(n)}\rangle - \langle m^{(n-1)}\rangle| \leq 2^{-1}\Gamma^{-1} 3^{-(n+2)(\tau+d+1)}. \quad (4.18)$$

Then,

$$|\langle m^{(n)}\rangle|^{-1} \leq \Gamma \prod_{k=0}^{\infty} \left(1 - 3^{-(k+2)(\tau+d+1)}\right)^{-1}. \quad (4.19)$$

Using the fact that $\prod(1-x_k)^{-1} = \exp(-\sum \ln(1-x_k))$, and that $-\ln(1-x) \leq x 2\ln 2$, $\forall x \in [0, 1/2]$, we show by straightforward calculations that $|\langle m^{(n)}\rangle|^{-1} \leq 2\Gamma$.
Thus we have proved (4.10)-(4.11) for all k.
We finally have to verify that with the choices (4.1)-(4.2),

the conditions (3.16)-(3.17) are satisfied at each step of the iteration. Equation (3.17) is guaranteed by Eq. (4.1): $ch_n^{-(\tau+d+1)} \geq ch^{-(\tau+d+1)} \geq 1$. For Eq. (3.16), we first notice that, since $\Gamma \geq 1$, $\Gamma_n \leq 2\Gamma$, and $ch^{-(\tau+d+1)} > 1$, we can write

$$\Gamma_{n-1}^3 c^2 h_n^{-2(\tau+d+1)} < 2^{-7}\gamma\delta^n.$$

Therefore, using Eq. (4.11) and Hypothesis (*iii*), we have

$$\Gamma_{n-1}^3 c^2 h_n^{-2(\tau+d+1)} V_{n-1} < 2^{-7}\left(\gamma\delta V_0\right)^{2^{n-1}} < \frac{1}{4}.$$

We have shown that, for sufficiently small initial perturbation, the Hamiltonian converges under successive iterations of the transformation to a Hamiltonian of the form (1.1) with $g = f = 0$. In order to complete the proof, we have still to verify that the composition of the infinitely many canonical transformations is a well-defined finite canonical transformation. It suffices to verify that $|\boldsymbol{\varphi}^{(\infty)} - \boldsymbol{\varphi}|$ and $|\boldsymbol{A}^{(\infty)} - \boldsymbol{A}|$ are finite. This is an immediate consequence of the fast convergence of the iteration. From Eq. (3.15), we deduce e. g. that

$$|\boldsymbol{\varphi}^{(\infty)} - \boldsymbol{\varphi}| \leq \sum_{n=1}^{\infty} h_n = 3h/2. \tag{4.20}$$

Analogously, using Eqs. (2.2), (3.3), (3.11), (3.12), and (4.11), for $|\boldsymbol{A}| \leq A_0$, we can bound

$$|\boldsymbol{A}^{(\infty)} - \boldsymbol{A}| < \text{const } \times \sum_{n=1}^{\infty} h_n. \tag{4.21}$$

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